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The cross-Wigner distribution as a generator of frames on the Euclidean plane from frames on the real line

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Abstract. Frames of square-integrable functions on the Euclidean plane are obtained from frames of square-integrable functions on the real line. This is achieved by computing the cross-Wigner distribution between elements of a frame for $L^2(R)$ and elements of the concomitant reciprocal frame. In addition to span $L^2(R^2)$, the functions constructed in this way verify an uncertainty principle, which makes them adequate for the analysis and representation of time–frequency distributions.

1. Introduction

Since the early introduction of the Wigner distribution within the context of quantum mechanics, several modifications to this concept have been advanced over the years. Historically, the main motive for modifying the Wigner distribution was the attempt to achieve a non-negative distribution. In a more general vein, the Wigner distribution was later adopted as a tool for the time–frequency analysis of quite *general* signals. Consequently, people wished for modifications whose goal was that of extracting the informational content that the signal (producing such a distribution) was assumed to convey [1, 2]. Among these modifications to the Wigner distribution one should mention those reported in [3–8]. The aim of such efforts was to produce distributions with some prescribed desirable properties.

At the present time it is clear that, in addition to the ability of defining 'informative' distributions for a given problem at hand, a proper time–frequency analysis entails the use of adequate methods for extracting information from such distributions. In this paper we introduce an approach specially devised for analysing and representing time–frequency distributions. We introduce a family of two-dimensional functions spanning $L^2(R^2)$ that can be used for analysing and representing arbitrary square-integrable functions on the Euclidean plane. The capability of these functions to span $L^2(R^2)$ is guaranteed by the fact that they constitute a frame for this space. The ingredients, out of which the functions are built up are: (a) a frame for $L^2(R)$ and the corresponding reciprocal frame and (b) the cross-Wigner distribution. We show that a frame for $L^2(R^2)$ is obtained by computing the cross-Wigner distribution between elements of a frame for $L^2(R)$ and the concomitant reciprocal ones. The thus constructed two-dimensional frame is endowed with the required feature of being adequate for the analysis and representation of time–frequency distributions, as the uncertainty principle becomes automatically verified.

The proposal has the additional advantage of being able to use all well known frames in *one dimension* for generating frames in *two dimensions*. Such two-dimensional frames, and the corresponding reciprocal ones, are easily obtained by computing Fourier transforms. The formalism is developed within the generalized frame structure [9–13] from which earlier proposed discrete frames [14, 15] arise as a particular case.

This paper is organized as follows. In section 2 we establish the notation and briefly review some frame properties which are relevant for our purposes. In section 3 we develop a formalism for generating two-dimensional frames. The present approach is illustrated by some examples in section 4. Finally, some conclusions are drawn in section 5.

2. A brief review on frames

Before advancing the frame definition let us introduce the notation to be used. Adopting Dirac's vector notation [16] we represent an element f of Hilbert space as a vector $|f\rangle$ and its dual as $\langle f|$.

Let $\mathcal{M}$ be a set of labels $\mathcal{M} = \{m \in \mathcal{M}\}$, μ a measure on $\mathcal{M}$ and $\mathcal{L}^2(\mu)$ that Hilbert space in which the identity operator reads

$$\hat{I}_{\mathcal{L}^2(\mu)} = \int_{\mathcal{M}} |m\rangle\langle m| \, \mathrm{d}\mu(m). \tag{1}$$

In the particular case for which $\mathcal{M} \equiv \mathcal{R}$ (the real line), m is a continuous parameter (say, t), and $\mathrm{d}\mu = \mathrm{d}t$, we shall denote $\mathcal{L}^2(\mu)$ as $\mathcal{H}_1$ and represent the inner product operation in $\mathcal{H}_1$ by $\langle\cdot|\cdot\rangle_{\mathcal{H}_1}$. Thus, for all $|f\rangle$ and $|g\rangle \in \mathcal{H}_1$, by inserting $\hat{I}_{\mathcal{H}_1} = \int_{\mathcal{R}} |t\rangle\langle t| \, \mathrm{d}t$ as in

$$\langle f|\hat{I}_{\mathcal{H}_1}|g\rangle_{\mathcal{H}_1} = \int_{\mathcal{R}} \langle f|t\rangle\langle t|g\rangle \, \mathrm{d}t = \int_{\mathcal{R}} \overline{f}(t)g(t) \, \mathrm{d}t \tag{2}$$

one is led to a representation of $\mathcal{H}_1$ in terms of the space of square-integrable functions on the real line. Indeed, through the set of δ-normalized continuous orthogonal vectors $\{|t\rangle\,;\ -\infty < t < \infty;\ \langle t|t'\rangle = \delta(t - t')\}$ the functional representation of elements of $\mathcal{H}_1$ is obtained as $\langle t|g\rangle = g(t)$ and $\langle g|t\rangle = \overline{\langle t|g\rangle} = \overline{g}(t)$, where $\overline{g}$ indicates the complex conjugate of g.

When $\mathcal{M} = \mathcal{R}^2$, $m = (t, \omega) \in \mathcal{R}^2$ and $\mathrm{d}\mu = \mathrm{d}t\,\mathrm{d}\omega$, we shall denote $\mathcal{L}^2(\mu)$ by $\mathcal{H}_2$ and represent the inner product operation in $\mathcal{H}_2$ by $\langle\cdot|\cdot\rangle_{\mathcal{H}_2}$. In such a case, for all Γ and $\Psi \in \mathcal{H}_2$, inserting $\hat{I}_{\mathcal{H}_2} = \int_{\mathcal{R}^2} |t, \omega\rangle\langle t, \omega| \, \mathrm{d}t\,\mathrm{d}\omega$ as in

$$\langle\Gamma|\hat{I}_{\mathcal{H}_2}|\Psi\rangle_{\mathcal{H}_2} = \int_{\mathcal{R}} \langle\Gamma|t, \omega\rangle\langle t, \omega|\Psi\rangle \, \mathrm{d}t\,\mathrm{d}\omega = \int_{\mathcal{R}} \overline{\Gamma}(t, \omega)\Psi(t, \omega) \, \mathrm{d}t\,\mathrm{d}\omega \tag{3}$$

one is led to a representation of $\mathcal{H}_2$ in terms of the space of square-integrable functions on the Euclidean plane, with $\langle t, \omega|t', \omega'\rangle = \delta(t - t')\delta(\omega - \omega')$, $\langle t, \omega|\Psi\rangle = \Psi(t, \omega)$ and $\langle\Gamma|t, \omega\rangle = \overline{\Gamma}(t, \omega)$.

In the next section we shall make use of the δ-distribution representation

$$\delta(t - t') = \int_{\mathcal{R}} \mathrm{e}^{-2\pi \mathrm{i}\omega(t-t')} \, \mathrm{d}\omega. \tag{4}$$

Given a set of labels $\mathcal{M} = \{m \in \mathcal{M}\}$ and a measure μ on $\mathcal{M}$, a family of vectors $|h_m\rangle \in \mathcal{H}_1$; $m \in \mathcal{M}$ is called a *generalized frame* [9, 13] (henceforth to be referred to simply as a frame) if, for every $|f\rangle \in \mathcal{H}_1$,

(a) the function $c(m) = \langle m|c\rangle = \langle h_m|f\rangle$ is measurable;

(b) there exists a pair of constants $0 < A \leqslant B < \infty$ such that

$$A\langle f|f\rangle_{\mathcal{H}_1} \leqslant \langle c|c\rangle_{\mathcal{L}^2(\mu)} \leqslant B\langle f|f\rangle_{\mathcal{H}_1}. \tag{5}$$

The constants A and B are called the frame bounds and (5) the frame condition. The latter implies that $|c\rangle \in \mathcal{L}^2(\mu)$ whenever $|f\rangle \in \mathcal{H}_1$. Thus the mapping $\hat{T} : \mathcal{H}_1 \mapsto \mathcal{L}^2(\mu)$ defines an operator,

$$\hat{T} = \int_{\mathcal{M}} |m\rangle\langle h_m| \, \mathrm{d}\mu(m) \tag{6}$$

and we have

$$|c\rangle = \hat{T}|f\rangle = \int_{\mathcal{M}} |m\rangle\langle h_m|f\rangle \, \mathrm{d}\mu(m) \tag{7}$$

$$\langle m'|c\rangle = \langle m'|\hat{T}|f\rangle = \langle h_{m'}|f\rangle. \tag{8}$$

The adjoint operator $\hat{T}^\dagger : \mathcal{L}^2(\mu) \mapsto \mathcal{H}_1$ is

$$\hat{T}^\dagger = \int_{\mathcal{M}} |h_m\rangle\langle m| \, \mathrm{d}\mu(m) \tag{9}$$

so that the frame condition can be expressed, in terms of the operator $\hat{G} = \hat{T}^\dagger\hat{T} : \mathcal{H}_1 \mapsto \mathcal{H}_1$, as

$$A\hat{I}_{\mathcal{H}_1} \leqslant \hat{G} \leqslant B\hat{I}_{\mathcal{H}_1}. \tag{10}$$

From (6) and (9) we see that $\hat{G}$ is given explicitly by

$$\hat{G} = \int_{\mathcal{M}} |h_m\rangle\langle h_m| \, \mathrm{d}\mu(m). \tag{11}$$

The inequality (10) entails that $\hat{G}$ has a bounded inverse $\hat{G}^{-1}$. In fact, $\hat{G}^{-1}$ satisfies [9, 10, 12, 15, 17]

$$B^{-1}\hat{I}_{\mathcal{H}_1} \leqslant \hat{G}^{-1} \leqslant A^{-1}\hat{I}_{\mathcal{H}_1}. \tag{12}$$

Assuming that $\hat{G}^{-1}$ is known explicitly, the reciprocal frame $\{|h^m\rangle; m \in \mathcal{M}\}$ is computed as $|h^m\rangle = \hat{G}^{-1}|h_m\rangle; m \in \mathcal{M}$. Thus, since $\hat{G}^{-1}\hat{G} = \hat{G}\hat{G}^{-1} = \hat{I}_{\mathcal{H}_1}$, by using (11) we obtain the following expression for the unity operator in $\mathcal{H}_1$:

$$\hat{I}_{\mathcal{H}_1} = \int_{\mathcal{M}} |h^m\rangle\langle h_m| \, \mathrm{d}\mu(m) = \int_{\mathcal{M}} |h_m\rangle\langle h^m| \, \mathrm{d}\mu(m). \tag{13}$$

The family $\{|h^m\rangle; m \in \mathcal{M}\}$ turns out to be a frame as well, with frame bounds B^{-1} and A^{-1} [9, 10, 12, 15, 17]. The reciprocal frame of $\{|h^m\rangle; m \in \mathcal{M}\}$ happens to be, again, the original frame [9, 15, 17]. When the frame bounds are equal, the frame is called a tight one, and the reciprocal frame satisfies $|h^m\rangle = |h_m\rangle/A; m \in \mathcal{M}$. For the case $A = 1$ the frame is self-reciprocal.

3. Generating two-dimensional frames

As already stated, an essential ingredient for building up two-dimensional frames is the cross-Wigner distribution. It is then pertinent to begin this section by recalling the corresponding definition.

Definition 1. *Given f and g in $\mathcal{H}_1$ the cross-Wigner distribution of f and g is defined as*

$$\Psi_{f,g}(t,\omega) = \int_{\mathcal{R}} \overline{f}(t - \tfrac{1}{2}x)\mathrm{e}^{-2\pi\mathrm{i}\omega x} g(t + \tfrac{1}{2}x)\,\mathrm{d}x \tag{14}$$

so that, in Dirac's notation we write

$$|\Psi_{f,g}\rangle = \int_{\mathcal{R}^2}\int_{\mathcal{R}} |t,\omega\rangle\langle f|t - \tfrac{1}{2}x\rangle \mathrm{e}^{-2\pi\mathrm{i}\omega x}\langle t + \tfrac{1}{2}x|g\rangle\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\omega \tag{15}$$

or, equivalently,

$$|\Psi_{f,g}\rangle = 2\int_{\mathcal{R}^2} |t,\omega\rangle\langle f|\hat{\Pi}_{t,\omega}|g\rangle_{\mathcal{H}_1}\,\mathrm{d}t\,\mathrm{d}\omega \qquad \langle\Psi_{f,g}| = 2\int_{\mathcal{R}^2}\langle g|\hat{\Pi}^{\dagger}_{t,\omega}|f\rangle_{\mathcal{H}_1}\langle t,\omega|\,\mathrm{d}t\,\mathrm{d}\omega \tag{16}$$

where $\hat{\Pi}_{t,\omega}$ denotes the unitary operator

$$\hat{\Pi}_{t,\omega} = \tfrac{1}{2}\int_{\mathcal{R}} |t - \tfrac{1}{2}x\rangle \mathrm{e}^{-2\pi\mathrm{i}\omega x}\langle t + \tfrac{1}{2}x|\,\mathrm{d}x \tag{17}$$

and $\hat{\Pi}^{\dagger}_{t,\omega}$ the adjoint of $\hat{\Pi}_{t,\omega}$, i.e.

$$\hat{\Pi}^{\dagger}_{t,\omega} = \tfrac{1}{2}\int_{\mathcal{R}} |t + \tfrac{1}{2}x\rangle \mathrm{e}^{2\pi\mathrm{i}\omega x}\langle t - \tfrac{1}{2}x|\,\mathrm{d}x. \tag{18}$$

From its definition it follows that the cross-Wigner distribution of $|f\rangle$ and $|g\rangle$ belongs to $\mathcal{H}_2$ for all $|f\rangle$ and $|g\rangle$ in $\mathcal{H}_1$. Indeed, from definition (15), and using (4), one has

$$\begin{aligned}\|\Psi_{f,g}\|^2 = \langle\Psi_{f,g}|\Psi_{f,g}\rangle_{\mathcal{H}_2} &= \int_{\mathcal{R}}\int_{\mathcal{R}}\langle g|t + \tfrac{1}{2}x\rangle\langle t + \tfrac{1}{2}x|g\rangle\langle f|t - \tfrac{1}{2}x\rangle\langle t - \tfrac{1}{2}x|f\rangle\,\mathrm{d}x\,\mathrm{d}t \\ &= \langle g|g\rangle_{\mathcal{H}_1}\langle f|f\rangle_{\mathcal{H}_1}.\end{aligned} \tag{19}$$

It is also straightforward to show that for all $|f\rangle$, $|g\rangle$, $|u\rangle$ and $|v\rangle$ in $\mathcal{H}_1$ the following relation holds:

$$\langle\Psi_{f,g}|\Psi_{u,v}\rangle_{\mathcal{H}_2} = \langle g|v\rangle_{\mathcal{H}_1}\langle u|f\rangle_{\mathcal{H}_1}. \tag{20}$$

Given a frame of vectors $|h_n\rangle \in \mathcal{H}_1; n \in \mathcal{M}$ and the corresponding reciprocal frame $|h^n\rangle = \hat{G}^{-1}|h_n\rangle; n \in \mathcal{M}$ we now define the cross-reciprocal vectors $|\Psi_{m,n}\rangle; n \in \mathcal{M}; m \in \mathcal{M}$ as

$$|\Psi_{m,n}\rangle = |\Psi_{h^m,h_n}\rangle = 2\int_{\mathcal{R}^2}|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}|h_n\rangle_{\mathcal{H}_1}\,\mathrm{d}t\,\mathrm{d}\omega \qquad m \in \mathcal{M} \quad n \in \mathcal{M}. \tag{21}$$

The next proposition shows that these vectors constitute a frame for $\mathcal{H}_2$.

Proposition 1. *Let $|h_n\rangle; n \in \mathcal{M}$ be a frame for $\mathcal{H}_1$ with frame bounds A, B. The cross-reciprocal vectors $|\Psi_{m,n}\rangle; m \in \mathcal{M}; n \in \mathcal{M}$ given in (21) constitute a frame for $\mathcal{H}_2$ with frame bounds A/B, B/A, i.e. the associated operator*

$$\hat{F} = \int_{\mathcal{M}^2}|\Psi_{m,n}\rangle\langle\Psi_{m,n}|\,\mathrm{d}\mu(m)\,\mathrm{d}\mu(n) \tag{22}$$

satisfies

$$\frac{A}{B}\hat{I}_{\mathcal{H}_2} \leqslant \hat{F} \leqslant \hat{I}_{\mathcal{H}_2}\frac{B}{A}. \tag{23}$$

Proof. Using both definition (21) and the fact that $|h_n\rangle = \hat{G}|h^n\rangle$; $n \in \mathcal{M}$, while $|h^m\rangle = \hat{G}^{-1}|h_m\rangle$; $m \in \mathcal{M}$, the operator $\hat{F}$ given in (22) can be recast in the form

$$\hat{F} = \int_{\mathcal{M}^2}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}\hat{G}|h^n\rangle_{\mathcal{H}_1}\langle h_n|\hat{\Pi}^{\dagger}_{t',\omega'}\hat{G}^{-1}|h_m\rangle_{\mathcal{H}_1} \times\langle t',\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}t'\,\mathrm{d}\omega'\,\mathrm{d}\mu(m)\,\mathrm{d}\mu(n). \tag{24}$$

Since $\int_{\mathcal{M}} |h^n\rangle\langle h_n|\,\mathrm{d}\mu(n)$ is a representation of the unity in $\mathcal{H}_1$, (24) becomes

$$\hat{F} = \int_{\mathcal{M}}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}\hat{G}\hat{\Pi}^{\dagger}_{t',\omega'}\hat{G}^{-1}|h_m\rangle_{\mathcal{H}_1}\langle t',\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}t'\,\mathrm{d}\omega'\,\mathrm{d}\mu(m). \tag{25}$$

By hypothesis, the operator $\hat{G}$ satisfies $A\hat{I}_{\mathcal{H}_1} \leqslant \hat{G} \leqslant B\hat{I}_{\mathcal{H}_1}$, while $\hat{G}^{-1}$ verifies $B^{-1}\hat{I}_{\mathcal{H}_1} \leqslant \hat{G} \leqslant A^{-1}\hat{I}_{\mathcal{H}_1}$. Hence, the operator $\hat{F}$ is bounded as follows:

$$\frac{A}{B}\int_{\mathcal{M}}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}\hat{\Pi}^{\dagger}_{t',\omega'}|h_m\rangle_{\mathcal{H}_1}\langle t',\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}t'\,\mathrm{d}\omega'\,\mathrm{d}\mu(m) \leqslant \hat{F} \leqslant \frac{B}{A}\int_{\mathcal{M}}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}\hat{\Pi}^{\dagger}_{t',\omega'}|h_m\rangle_{\mathcal{H}_1}\langle t',\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}t'\,\mathrm{d}\omega'\,\mathrm{d}\mu(m). \tag{26}$$

The proof is completed below by showing that the integral appearing in the bounds given above is a representation of the unity operator in $\mathcal{H}_2$. Indeed,

$$\langle h^m|\hat{\Pi}_{t,\omega}\hat{\Pi}^{\dagger}_{t',\omega'}|h_m\rangle = \tfrac{1}{4}\int_{\mathcal{R}}\int_{\mathcal{R}}\langle h^m|t-\tfrac{1}{2}x\rangle\delta(t+\tfrac{1}{2}x-t'-\tfrac{1}{2}x') \times\mathrm{e}^{-2\pi\mathrm{i}\omega x}\mathrm{e}^{2\pi\mathrm{i}\omega' x'}\langle t'-\tfrac{1}{2}x'|h_m\rangle\,\mathrm{d}x\,\mathrm{d}x' \tag{27}$$

and, since $\delta(t+\frac{1}{2}x-t'-\frac{1}{2}x') = 2\delta(x'-2t-x+2t')$, we have

$$\langle h^m|\hat{\Pi}_{t,\omega}\hat{\Pi}^{\dagger}_{t',\omega'}|h_m\rangle = \tfrac{1}{2}\int_{\mathcal{R}}\langle h^m|t-\tfrac{1}{2}x\rangle\langle 2t'-t-\tfrac{1}{2}x|h_m\rangle\mathrm{e}^{-2\pi\mathrm{i}\omega x}\mathrm{e}^{2\pi\mathrm{i}\omega'(2t+x-2t')}\,\mathrm{d}x. \tag{28}$$

Thus, by using the fact that $\int_{\mathcal{M}}|h_m\rangle\langle h^m|\,\mathrm{d}\mu(m) = \hat{I}_{\mathcal{H}_1}$, we finally obtain

$$\begin{aligned}
&\int_{\mathcal{M}}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}\hat{\Pi}^{\dagger}_{t',\omega'}|h_m\rangle_{\mathcal{H}_1}\langle t',\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}t'\,\mathrm{d}\omega'\,\mathrm{d}\mu(m)\\
&\quad = 2\int_{\mathcal{R}^4}\int_{\mathcal{R}}|t,\omega\rangle\delta(2t-2t')\mathrm{e}^{-2\pi\mathrm{i}\omega x}\mathrm{e}^{2\pi\mathrm{i}\omega'(2t+x-2t')}\langle t',\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}t'\,\mathrm{d}\omega\,\mathrm{d}x\\
&\quad = \int_{\mathcal{R}^2}\int_{\mathcal{R}}\int_{\mathcal{R}}\mathrm{e}^{-2\pi\mathrm{i}x(\omega-\omega')}|t,\omega\rangle\langle t,\omega'|\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}x\\
&\quad = \int_{\mathcal{R}^2}|t,\omega\rangle\langle t,\omega|\,\mathrm{d}t\,\mathrm{d}\omega = \hat{I}_{\mathcal{H}_2}.
\end{aligned} \tag{29}$$

□

We now show that the set of vectors $|\Psi_{h_m,h^n}\rangle$; $n \in \mathcal{M}$; $m \in \mathcal{M}$ gives rise to the corresponding reciprocal frame.

Proposition 2. *The reciprocal frame* $|\Psi^{m,n}\rangle = \hat{F}^{-1}|\Psi_{m,n}\rangle$; $n \in \mathcal{M}$; $m \in \mathcal{M}$ *can be computed as*

$$|\Psi^{m,n}\rangle \equiv |\Psi_{h_m,h^n}\rangle = 2\int_{\mathcal{R}^2}|t,\omega\rangle\langle h_m|\hat{\Pi}_{t,\omega}|h^n\rangle_{\mathcal{H}_1}\,\mathrm{d}t\,\mathrm{d}\omega \qquad n \in \mathcal{M} \quad m \in \mathcal{M}. \tag{30}$$

We prove this proposition by showing that *(a) The vectors $|\Psi^{m,n}\rangle$; $n \in \mathcal{M}$; $m \in \mathcal{M}$ given in (30) give rise to a representation of the unity operator in $\mathcal{H}_2$, i.e.*

$$\hat{I}_{\mathcal{H}_2} = \int_{\mathcal{M}^2} |\Psi^{m,n}\rangle\langle\Psi_{m,n}| \, \mathrm{d}\mu(m)\, \mathrm{d}\mu(n) = \int_{\mathcal{M}^2} |\Psi_{m,n}\rangle\langle\Psi^{m,n}| \, \mathrm{d}\mu(m)\, \mathrm{d}\mu(n).$$

(b) $$\langle\Psi_{l,k}|\Psi^{m,n}\rangle_{\mathcal{H}_2} = \langle\Psi^{l,k}|\Psi_{m,n}\rangle_{\mathcal{H}_2}.$$

(c) $$\hat{F}|\Psi^{m,n}\rangle = |\Psi_{m,n}\rangle.$$

The proof of (a) is straightforward since, by explicitly writing $|\Psi_{m,n}\rangle$ as given in (21), and $\langle\Psi^{m,n}|$ as the dual of (30), one has

$$\begin{aligned}\int_{\mathcal{M}^2} |\Psi_{m,n}\rangle\langle\Psi^{m,n}| \, \mathrm{d}\mu(m)\, \mathrm{d}\mu(n) &= \int_{\mathcal{M}^2}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}|h_n\rangle_{\mathcal{H}_1}\langle h^n|\hat{\Pi}^\dagger_{t',\omega'}|h_m\rangle_{\mathcal{H}_1} \\ &\quad\times\langle t',\omega'| \, \mathrm{d}t\, \mathrm{d}\omega\, \mathrm{d}t'\, \mathrm{d}\omega'\, \mathrm{d}\mu(m)\, \mathrm{d}\mu(n) \\ &= \int_{\mathcal{M}}\int_{\mathcal{R}^4} 4|t,\omega\rangle\langle h^m|\hat{\Pi}_{t,\omega}\hat{\Pi}^\dagger_{t',\omega'}|h_m\rangle_{\mathcal{H}_1}\langle t',\omega'| \, \mathrm{d}t\, \mathrm{d}\omega\, \mathrm{d}t'\, \mathrm{d}\omega'\, \mathrm{d}\mu(m) \qquad (31)\end{aligned}$$

which proves (a) (cf equation (29)).

To prove (b) let us recall that the notation $|\Psi_{l,k}\rangle$ actually means $|\Psi_{h^l,h_k}\rangle$, while $|\Psi^{m,n}\rangle$ stands for $|\Psi_{h_m,h^n}\rangle$. Thus, property (20), with $f = h^l$, $g = h_k$, $u = h_m$, $v = h^n$, implies

$$\begin{aligned}\langle\Psi_{l,k}|\Psi^{m,n}\rangle_{\mathcal{H}_2} &= \langle\Psi_{h^l,h_k}|\Psi_{h_m,h^n}\rangle_{\mathcal{H}_2} = \langle h_k|h^n\rangle_{\mathcal{H}_1}\langle h_m|h^l\rangle_{\mathcal{H}_1} \\ &= \langle h_k|\hat{G}^{-1}|h_n\rangle_{\mathcal{H}_1}\langle h_m|\hat{G}^{-1}|h_l\rangle_{\mathcal{H}_1} = \langle h^k|h_n\rangle_{\mathcal{H}_1}\langle h^m|h_l\rangle_{\mathcal{H}_1}. \qquad (32)\end{aligned}$$

On the other hand,

$$\langle\Psi^{l,k}|\Psi_{m,n}\rangle_{\mathcal{H}_2} = \langle\Psi_{h_l,h^k}|\Psi_{h^m,h_n}\rangle_{\mathcal{H}_2} = \langle h^k|h_n\rangle_{\mathcal{H}_1}\langle h^m|h_l\rangle_{\mathcal{H}_1} \qquad (33)$$

which proves (b).

By using first (b) and then (a) we are now in a position to readily prove (c)

$$\begin{aligned}\hat{F}|\Psi^{m,n}\rangle &= \int_{\mathcal{M}^2} |\Psi_{l,k}\rangle\langle\Psi_{l,k}|\Psi^{m,n}\rangle_{\mathcal{H}_2} \, \mathrm{d}\mu(l)\, \mathrm{d}\mu(k) \\ &= \int_{\mathcal{M}^2} |\Psi_{l,k}\rangle\langle\Psi^{l,k}|\Psi_{m,n}\rangle_{\mathcal{H}_2} \, \mathrm{d}\mu(l)\, \mathrm{d}\mu(k) = |\Psi_{m,n}\rangle. \qquad (34)\end{aligned}$$

Corollary 1. *Every vector $|\Psi\rangle \in \mathcal{H}_2$ admits of an expansion of the form*

$$|\Psi\rangle = \int_{\mathcal{M}^2} D(m,n)|\Psi_{m,n}\rangle \, \mathrm{d}\mu(m)\, \mathrm{d}\mu(n) \qquad (35)$$

with

$$D(m,n) = \langle\Psi^{m,n}|\Psi\rangle_{\mathcal{H}_2} = 2\int_{\mathcal{R}^2} \langle h^n|\hat{\Pi}^\dagger_{t,\omega}|h_m\rangle_{\mathcal{H}_1}\langle t,\omega|\Psi\rangle \, \mathrm{d}t\, \mathrm{d}\omega \qquad (36)$$

and an expansion of the form

$$|\Psi\rangle = \int_{\mathcal{M}^2} D(m,n)|\Psi^{m,n}\rangle \, \mathrm{d}\mu(m)\, \mathrm{d}\mu(n) \qquad (37)$$

with

$$D(m,n) = \langle\Psi_{m,n}|\Psi\rangle_{\mathcal{H}_2} = 2\int_{\mathcal{R}^2} \langle h_n|\hat{\Pi}^\dagger_{t,\omega}|h^m\rangle\langle t,\omega|\Psi\rangle \, \mathrm{d}t\, \mathrm{d}\omega. \qquad (38)$$

The proof follows from part (a) of proposition 2 and the identity $|\Psi\rangle = \hat{I}_{\mathcal{H}_2}|\Psi\rangle$.

4. Examples

4.1. A two-dimensional Gabor-like transform

Let $\mathcal{M} = \mathcal{R}^2$ be the set of all continuous parameters $m = (\alpha, \beta)$ and $d\mu(m) = d\alpha\, d\beta$. Let us further consider that the frame elements $|h_m\rangle \equiv |h_{\alpha,\beta}\rangle$ are the Weyl–Heisenberg coherent states [17–20], so that the functional representation of $|h_{\alpha,\beta}\rangle$ is given by

$$\langle t|h_{\alpha,\beta}\rangle = h_{\alpha,\beta}(t) = h(t-\alpha)\,e^{i2\pi\beta t} \tag{39}$$

with $h(t)$ any function in $\mathcal{H}_1$, normalized to unity. The functions $h(t-\alpha)\,e^{i2\pi\beta t}$ are also called Gabor functions because they generate the so-called Gabor transform, which is defined as $\langle h_{\alpha,\beta}|f\rangle_{\mathcal{H}_1}\ \forall\,|f\rangle \in \mathcal{H}_1$. For all $(\alpha, \beta) \in \mathcal{R}^2$ the frame (39) is known to be self-reciprocal, as the reciprocal frame happens to be $|h^{\alpha\beta}\rangle \equiv |h_{\alpha,\beta}\rangle\ \forall(\alpha,\beta) \in \mathcal{R}^2$ [9, 13]. Thus, the two-dimensional frame that one obtains from (21) is

$$\begin{aligned}\Psi_{\alpha\beta\alpha'\beta'}(t,\omega) &= \langle t,\omega|\Psi_{\alpha\beta\alpha'\beta'}\rangle_{\mathcal{H}_1}\\ &= \int_{\mathcal{R}} \overline{h}(t-\tfrac{1}{2}x-\alpha)h(t+\tfrac{1}{2}x-\alpha')\,e^{-it2\pi(\beta-\beta')}e^{-i2\pi\omega x}e^{i\pi x(\beta+\beta')}\,dx\end{aligned} \tag{40}$$

and it is also self-reciprocal, since $|\Psi^{\alpha\beta\alpha'\beta'}\rangle \equiv |\Psi_{\alpha\beta\alpha'\beta'}\rangle$. For the special case $h(t) = (2\sigma)^{1/4}e^{-\sigma\pi t^2}$, the integral in (40) can be evaluated analytically. It turns out to be

$$\Psi_{\alpha\beta\alpha'\beta'}(t,\omega) = 2e^{-2\pi\sigma(t-(\alpha+\alpha')/2)^2}e^{-(2\pi/\sigma)(\omega-(\beta+\beta')/2)^2}e^{2\pi i\omega(\alpha-\alpha')}e^{-\pi i(\alpha-\alpha')(\beta+\beta')}e^{-2\pi it(\beta-\beta')}. \tag{41}$$

It should be stressed that, as a consequence of the fact that the proposed two-dimensional frame is obtained by recourse to Fourier transforms, its elements verify an uncertainty principle. This is clearly seen in the analytic example (41). Note that the peak of one of the Gaussians (that over the variable ω) becomes less pronounced as the parameter σ increases, whereas the other Gaussian, as a function of t, exhibits a 'sharper' peak. On the other hand, by decreasing σ the inverse effect takes place.

The frame elements (40), and the particular ones (41), give rise to a Gabor-like transform for two-dimensional functions $\Gamma \in \mathcal{H}_2$, which is computed as

$$C_{\alpha,\beta,\alpha',\beta'} = \langle\Psi^{\alpha,\beta,\alpha',\beta'}|\Gamma\rangle_{\mathcal{H}_2} = \langle\Psi_{\alpha,\beta,\alpha',\beta'}|\Gamma\rangle_{\mathcal{H}_2} = \int_{\mathcal{R}^2}\overline{\Psi}_{\alpha,\beta,\alpha',\beta'}(t,\omega)\Gamma(t,\omega)\,dt\,d\omega. \tag{42}$$

Corollary 1 entails that the function $\Gamma(t,\omega)$ can be recovered from all the $C_{\alpha,\beta,\alpha',\beta'}$ values as

$$\Gamma(t,\omega) = \int_{\mathcal{R}^2}\int_{\mathcal{R}^2} C_{\alpha,\beta,\alpha',\beta'}\Psi_{\alpha,\beta,\alpha',\beta'}(t,\omega)\,d\alpha\,d\beta\,d\alpha'\,d\beta'. \tag{43}$$

4.2. The discrete case

Let us consider now that $\mathcal{M} = \mathcal{Z}$ (the set of integer numbers), where μ is the counting measure on $\mathcal{Z}$ and $\mathcal{L}^2(\mu) = l^2(\mathcal{Z})$ (the space of all square-summable sequences). Let $|h_m\rangle$; $m \in \mathcal{Z}$ be a discrete frame for $\mathcal{H}_1$ with the corresponding reciprocal frame $|h^m\rangle$; $m \in \mathcal{Z}$. The proposed discrete frame $|\Psi_{m,n}\rangle$ for $\mathcal{H}_2$ is thereby computed as in (21), and the corresponding reciprocal frame as in (30). According to corollary 1, every $\Gamma \in \mathcal{H}_2$ admits an expansion of the form

$$|\Gamma\rangle = \sum_{n\in\mathcal{Z}}\sum_{m\in\mathcal{Z}} C_{m,n}|\Psi_{m,n}\rangle \tag{44}$$

with

$$C_{m,n} = \langle \Psi^{m,n} | \Gamma \rangle_{\mathcal{H}_2} = \int_{\mathcal{R}^2} \int_{\mathcal{R}} \overline{h}^n (t + \tfrac{1}{2}x) h_m (t - \tfrac{1}{2}x)\, \mathrm{e}^{2\pi \mathrm{i} \omega x} \Gamma(t, \omega)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}\omega. \tag{45}$$

Unless the frame $|h_m\rangle; m \in \mathcal{M}$ is a tight one, its reciprocal, $|h^m\rangle; m \in \mathcal{M}$, has to be computed by recourse to iterative algorithms [17, 21]. Since, in the discrete case, a normalized tight frame with frame bounds $A = B = 1$ corresponds to an orthonormal basis, it may appear that, according to (23), a normalized tight frame for $\mathcal{H}_1$ gives rise to an orthogonal basis for $\mathcal{H}_2$. Actually, if we consider $A = B$ in (23) the frame bounds for $|\Psi_{m,n}\rangle; n \in \mathcal{Z}; m \in \mathcal{Z}$ turn out be given by unity. However, $|\Psi_{m,n}\rangle; n \in \mathcal{Z}; m \in \mathcal{Z}$ is not an orthonormal basis because it follows from (19) that if $\|h_m\|^2 = 1; m \in \mathcal{Z}$, then $\|\Psi_{m,n}\|^2 \neq 1; n \in \mathcal{Z}; m \in \mathcal{Z}$. Indeed, if $|h_m\rangle; m \in \mathcal{Z}$ is a tight frame, then $|h^m\rangle = |h_m\rangle / A$, so that $\|h_m\|^2 = 1; m \in \mathcal{Z}$ implies $\|h^m\|^2 = 1/A^2; m \in \mathcal{Z}$ and, from (19), we gather that $\|\Psi_{m,n}\|^2 = \|h^m\|^2 \|h_n\|^2 = 1/A^2; n \in \mathcal{Z}; m \in \mathcal{Z}$. Hence, $\|\Psi_{m,n}\|^2 \neq 1$ unless $A = 1$. If, in order to renormalize the vectors $|\Psi_{m,n}\rangle$, we define the new vectors $|\tilde{\Psi}_{m,n}\rangle = A|\Psi_{m,n}\rangle$, these new vectors are now normalized to unity, but they constitute a frame with new bounds, namely, A^3/B, AB. This discussion leads one to conclude that if $|h_m\rangle; m \in \mathcal{Z}$ is a tight frame for $\mathcal{H}_1$, the cross reciprocal vectors (21) constitute a tight frame for $\mathcal{H}_2$. For these vectors to be an orthonormal basis for $\mathcal{H}_2$, the vectors $|h_m\rangle; m \in \mathcal{Z}$ must constitute an orthonormal basis for $\mathcal{H}_1$.

5. Conclusions

A method for generating frames on the Euclidean plane from frames on the real line has been advanced. It was shown that, by computing the cross-Wigner distribution of frames elements in $\mathcal{H}_1 = L^2(\mathcal{R})$ and the concomitant reciprocal ones, a family of vectors constituting a frame for $\mathcal{H}_2 = L^2(\mathcal{R}^2)$ is obtained. Since these vectors span $\mathcal{H}_2$ they can be used for representing any element of such a space. Furthermore, since the vectors are obtained by computing Fourier transforms, their functional representation verifies the uncertainty principle, which makes them adequate for analysing time–frequency (or coordinate–momentum) distributions. The reciprocal frame in $\mathcal{H}_2$ is computed by recourse to a technique similar to that for computing the frame. Thus, all well known frames in $\mathcal{H}_1$ can be used for generating frames in $\mathcal{H}_2$ by means of a simple computational task that amounts to computing Fourier transforms. In particular, if one considers the frame in $\mathcal{H}_1$ to be the Weyl–Heisenberg coherent states, our technique yields a two-dimensional Gabor-like transform.

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